§3.9 Antiderivatives (Anti-D)

Key points: ① Definition of anti-D and the most general anti-D of $f(x)$.

② Finding anti-D using derivative table and linear rules.

③ Velocity and position as anti-D.

Definition: If $F'(x) = f(x)$, then $F(x)$ is ONE ANTIDERIVATIVE of $f(x)$.

$F(x)+C$ is called THE MOST GENERAL ANTIDERIVATIVE of $f(x)$, where C is arbitrary constant.

e.g. 1. $(x^2)' = 2x$, $2x$ is the derivative of x^2; x^2 is one anti-D of $2x$.

$(x^2+5)' = 2x$, $2x$ is the derivative of x^2+5; x^2+5 is one anti-D of $2x$.

For any constant C, $(x^2+C)' = 2x$, $2x$ is the derivative of x^2+C, x^2+C is one anti-D of $2x$.

x^2+C is called the most general anti-D of $2x$.

• (Anti) derivative table:

	$F(x)$	$f(x) = F'(x)$		
☆	x^n	$n \cdot x^{n-1}$	$n\cdot x^{n-1}$ has anti-D x^n	$\boxed{x^n \text{ has anti-D } \frac{1}{n+1}x^{n+1}}$ $(n \neq -1)$
	$\sin x$	$\cos x$	$\cos x$ has anti-D $\sin x$.	$\cos x$ has anti-D $\sin x$
	$\cos x$	$-\sin x$	$-\sin x$ has anti-D $\cos x$	$\boxed{\sin x \text{ has anti-D } -\cos x}$
	$\tan x$	$\sec^2 x$.	$\sec^2 x$ has anti-D $\tan x$	
	$\sec x$	$\tan x \cdot \sec x$.	$\tan x \sec x$ has anti-D $\sec x$.	

• Linear rule: If $f(x)$ has anti-D $F(x)$, $g(x)$ has anti-D $G(x)$, then

$$a\cdot f(x) + b\cdot g(x) \text{ has anti-D } a F(x) + b\cdot G(x)$$

eg. 2. Find one anti-D of (a) $f(x) = 2x^5$, (b), $f(x) = \frac{\sin x}{2}$, (c) $f(x) = 2x^5 + \frac{\sin x}{2}$.

(a): $f(x) = 2\cdot\frac{1}{6}\cdot\boxed{6x^5}$. Notice $(x^6)' = 6\cdot x^5 \Rightarrow$ anti-D of $f(x)$ is $F(x) = 2\cdot\frac{1}{6}\cdot x^6$

(b): $f(x) = \frac{\sin x}{2} = \frac{(-1)}{2}\cdot(-\sin x)$. $(\cos x)' = -\sin x \Rightarrow$ anti-D of $f(x)$ is $F(x) = \frac{-1}{2}\cdot\cos x$.

(c) According to (a), (b). $2x^5 + \frac{\sin x}{2}$ has one anti-D $2\cdot\frac{1}{6}x^6 + \frac{-1}{2}\cos x$.

☆ Key anti-D formula: $\boxed{x^n \xrightarrow{\text{anti-D}} \frac{1}{n+1}x^{n+1}}$, $n \neq -1$

eg.3 Find one anti-DF for the following functions f:

(a): $f(x)=1 \Rightarrow F(x)=x$; (a'): $f(x)=-\frac{1}{3} \Rightarrow F(x)=-\frac{1}{3}\cdot x$. (formula with $n=0$)

(b): $f(x)=5x^1 \Rightarrow F(x)=5\cdot\frac{1}{1+1}\cdot x^{1+1}=5\cdot\frac{1}{2}x^2$ (formula with $n=1$)

(c): $f(t)=t^3 \Rightarrow F(t)=\frac{1}{3+1}\cdot t^{3+1}=\frac{1}{4}\cdot t^4$ (formula with $n=3$)

Remark: The formula is also applied to negative n and fraction n.

☆ (d): $f(x)=\frac{1}{x^2} \Rightarrow f(x)=x^{-2} \Rightarrow F(x)=\frac{1}{-2+1}\cdot x^{-2+1}=-x^{-1}=\frac{-1}{x}$ ($n=-2$)

☆ (e): $f(t)=2\sqrt{t} \Rightarrow f(t)=2\cdot t^{\frac{1}{2}} \Rightarrow F(t)=2\cdot\frac{1}{\frac{1}{2}+1}\cdot t^{\frac{1}{2}+1}=2\cdot\frac{1}{\frac{3}{2}}\cdot t^{\frac{3}{2}}=\frac{4}{3}\cdot t^{\frac{3}{2}}$ ($n=\frac{1}{2}$)

☆ According to the definition of Anti-D, the (most general) anti-D of $f'(x)$ is $f(x)+C$. With extra condition on $f(x)$, we can determine the value of C.

eg4. Suppose $f'(x)=\sin x$ and $f(\frac{\pi}{2})=0$. Find $f(x)$.

Solution: $f(x)$ is the anti-D of $f'(x)=\sin x$. Therefore, $f(x)=-\cos x+C$

Furthermore, plug $x=\frac{\pi}{2}$ into $f(x)=-\cos x+C$, we have,

$$f(\tfrac{\pi}{2})=-\cos\tfrac{\pi}{2}+C \Leftrightarrow 0=-0+C \quad \text{since } f(\tfrac{\pi}{2})=0,\ \cos\tfrac{\pi}{2}=0$$

$$\Rightarrow C=0 \quad \text{plug into } f(x)=-\cos x+C.$$

$$\boxed{f(x)=-\cos x}$$

Remark: When you get the expression for $f(x)$, it is convinient to double check your answer by computing $f'(x)$ and $f(\frac{\pi}{2})$.

• Moving particle. Position: $S(t)$ Velocity: $V(t)$ Acceleration: $a(t)$

Relation: $S'(t)=V(t)$, $V'(t)=a(t)$

$S(t)$ is the anti-D of $V(t)$; $V(t)$ is the anti-D of $a(t)$

Related problems: Give $V(t)$, find $S(t)$. Give $a(t)$, find $V(t)$.

e.g. 5. A particle is moving along a line with acceleration given by $a(t)=4t^3+2\sin t$.
(f16). Given the initial ~~value~~ velocity is $V(0)=5$ m/s, find the velocity
at time $t=\pi$ seconds.

Hint: V is the (general) anti-D of $a(t)$. Find the general anti-D of $4t^3+2\sin t$. Then use the initial condition to determine the constant C

Solution: $t^3 \xrightarrow{\text{anti-D}} \frac{1}{3+1} t^{3+1} = \frac{1}{4} t^4$ $(n=3)$; $\sin t \xrightarrow{\text{anti-D}} -\cos t$.

The general anti-D of $a(t)=4t^3+2\sin t$ is $4(\frac{1}{4}t^4)+2(-\cos t)+C$

i.e. $\boxed{V(t)} = 4(\frac{1}{4}t^4)+2(-\cos t)+C = \boxed{t^4-2\cos t+C}$

Plug in $t=0$: $5=V(0)=0^4-2\cos 0+C=0-2+C$ since $\cos 0=1$.

$\Rightarrow 5=-2+C \Rightarrow C=7$. Plug back into V's expression.

$$\boxed{V(t)=t^4-2\cos t+7}$$

Then evaluate V at $t=\pi$, i.e., $\boxed{\begin{aligned} V(\pi)&=\pi^4-2\cdot\cos\pi+7; \\ &=\pi^4+9 \text{ m/s}\end{aligned}}$ $\cos\pi=-1$.

Hints for WW.

※3. Rewrite $f(x)=\frac{7-5x^9}{x^3}=\frac{7}{x^3}-\frac{5x^9}{x^3}=7\cdot x^{-3}-5\cdot x^6$

※4. $y=f(x)$ goes through $(1,0)$ means $f(1)=0$.
The slope of the tangent line $=f'(x)=\frac{6}{x^3}-\frac{9}{x^5}=6x^{-3}-9\cdot x^{-5}$
Then use the method in e.g. 4 to find $f(x)$.

※5. $\frac{1}{\sqrt[3]{x}}=x^{-\frac{1}{3}}$, $\sqrt[3]{x^2}=x^{\frac{2}{3}}$. Apply the anti-D formula with $n=-\frac{1}{3}$ and $n=\frac{2}{3}$

※6. $(x+4)^n$ has anti-D $\frac{1}{n+1}\cdot(x+4)^{n+1}$. For example, $(x+4)^3 \xrightarrow{\text{anti-D}} \frac{1}{4}(x+4)^4$.
$(x+4)^4 \xrightarrow{\text{anti-D}} \frac{1}{5}(x+4)^5$

※7. $1 \text{ mph} = \frac{1 \text{ mile}}{1 \text{ hour}} = \frac{22}{15}$ ft/second.
Decelerate at 26 ft/s means $a(t)=-26$ ft/s² $\Rightarrow V(t)=-26\cdot t$ ft/s